

Laguerre Geometry of Hypersurfaces in \mathbb{R}^n

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Abstract

Laguerre geometry of surfaces in \mathbb{R}^3 is given in the book of Blaschke [1], and have been studied by E.Musso and L.Nicolodi [5], [6], [7], B. Palmer [8] and other authors. In this paper we study Laguerre differential geometry of hypersurfaces in \mathbb{R}^n . For any umbilical free hypersurface $x : M \rightarrow \mathbb{R}^n$ with non-zero principal curvatures we define a Laguerre invariant metric g on M and a Laguerre invariant self-adjoint operator $\mathbb{S} : TM \rightarrow TM$, and show that $\{g, \mathbb{S}\}$ is a complete Laguerre invariant system for hypersurfaces in \mathbb{R}^n with $n \geq 4$. We calculate the Euler-Lagrange equation for the Laguerre volume functional of Laguerre metric by using Laguerre invariants. Using the Euclidean space \mathbb{R}^n , the Lorentzian space \mathbb{R}_1^n and the degenerate space \mathbb{R}_0^n we define three Laguerre space forms $U\mathbb{R}^n$, $U\mathbb{R}_1^n$ and $U\mathbb{R}_0^n$ and define the Laguerre embedding $U\mathbb{R}_1^n \rightarrow U\mathbb{R}^n$ and $U\mathbb{R}_0^n \rightarrow U\mathbb{R}^n$, analogue to the Moebius geometry where we have Moebius space forms S^n , \mathbb{H}^n and \mathbb{R}^n (spaces of constant curvature) and conformal embedding $\mathbb{H}^n \rightarrow S^n$ and $\mathbb{R}^n \rightarrow S^n$ (cf. [4], [10]). Using these Laguerre embedding we can unify the Laguerre geometry of hypersurfaces in \mathbb{R}^n , \mathbb{R}_1^n and \mathbb{R}_0^n . As an example we show that minimal surfaces in \mathbb{R}_1^3 or \mathbb{R}_0^3 are Laguerre minimal in \mathbb{R}^3 .

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§ 1. Introduction

In study of contact structure in the unit tangent bundle US^n over unit sphere S^n Sophus Lie discovered a interesting finite dimensional transformation group $LT\mathbb{G}$, which preserves oriented $(n-1)$ -spheres in US^n . This group $LT\mathbb{G}$ is called Lie sphere transformation group, which is isomorphic to the group $O(n+1, 2)/\{\pm 1\}$, where $O(n+1, 2)$ is the Lorentzian group in the Lorentzian space \mathbb{R}_2^{n+3} . There are two interesting types of subgroups of $LT\mathbb{G}$, one is called Moebius group $M\mathbb{G}$, consisting of all elements of $O(n+1, 2)$ which fix a time-like vector in \mathbb{R}_2^{n+3} ; another is called Laguerre group $L\mathbb{G}$, consisting of all elements of $O(n+1, 2)$ which fix a light-like vector in \mathbb{R}_2^{n+3} .

In Laguerre differential geometry we study invariants of hypersurfaces in Euclidean space \mathbb{R}^n under the Laguerre transformation group.

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Laguerre geometry of surfaces in \mathbb{R}^3 is given in the book of W.Blaschke [1], and have been studied by E.Musso and L.Nicolodi [5], [6] and other authors.

In this paper we study Laguerre differential geometry of hypersurfaces in \mathbb{R}^n . For any umbilical-free hypersurface $x : M \rightarrow \mathbb{R}^n$ with non-zero principal curvatures we define a Laguerre invariant metric g on M and a Laguerre invariant self-adjoint operator $\mathbb{S} : TM \rightarrow TM$, and show that $\{g, \mathbb{S}\}$ is a complete Laguerre invariant system for hypersurfaces in \mathbb{R}^n with $n \geq 4$. We calculate the Euler-Lagrange equation for the Laguerre volume functional by using Laguerre invariants.

Using \mathbb{R}^n , Lorentzian space \mathbb{R}_1^n and degenerate space \mathbb{R}_0^n corresponding to the space-like hyperplane, Lorentzian hyperplane and degenerate hyperplane in \mathbb{R}_1^{n+1} we define three Laguerre space forms $U\mathbb{R}^n$, $U\mathbb{R}_1^n$ and $U\mathbb{R}_0^n$ as suitable bundle over \mathbb{R}^n , \mathbb{R}_1^n , \mathbb{R}_0^n and define the Laguerre embedding $U\mathbb{R}_1^n \rightarrow U\mathbb{R}^n$ and $U\mathbb{R}_0^n \rightarrow U\mathbb{R}^n$, analogue to the Moebius geometry where we have Moebius space forms S^n , \mathbb{H}^n and \mathbb{R}^n (spaces of constant curvature) and conformal embedding $\mathbb{H}^n \rightarrow S^n$ and $\mathbb{R}^n \rightarrow S^n$ (cf. [4], [10]). Using these Laguerre embedding we can unify the Laguerre geometries of hypersurfaces in \mathbb{R}^n , \mathbb{R}_1^n and \mathbb{R}_0^n . As an example we show that minimal surfaces in \mathbb{R}_1^3 or \mathbb{R}_0^3 are Laguerre minimal in \mathbb{R}^3 .

We organize the paper as follows. In §2 we study the geometry of oriented spheres in \mathbb{R}^n . In §3 we study Laguerre transformation group on $U\mathbb{R}^n$. In §4 we define Laguerre space forms and Laguerre embedding. In §5 and §6 we study Laguerre invariants for hypersurfaces in \mathbb{R}^n and prove the fundamental theorem. In §7 we calculate Euler-Lagrange equation for volume function of Laguerre metric. In §8 we unify the geometry of Laguerre hypersurfaces in \mathbb{R}_1^n , \mathbb{R}_0^n and \mathbb{R}^n .

§ 2. Geometry of oriented spheres in \mathbb{R}^n

Let $U\mathbb{R}^n$ be the unit tangent bundle over \mathbb{R}^n , which is the hypersurface in \mathbb{R}^{2n} :

$$(2.1) \quad U\mathbb{R}^n = \{(x, \xi) \mid x \in \mathbb{R}^n, \xi \in S^{n-1}\} = \mathbb{R}^n \times S^{n-1} \subset \mathbb{R}^{2n}.$$

An oriented sphere in $U\mathbb{R}^n$ centered at p with radius r is the $(n-1)$ -dimensional submanifold in $U\mathbb{R}^n$ given by

$$(2.2) \quad S(p, r) = \{(x, \xi) \in U\mathbb{R}^n \mid x - p = r\xi\}.$$

Geometrically, $S(p, r)$ with $r \neq 0$ corresponds to the oriented sphere in \mathbb{R}^n centered at $p \in \mathbb{R}^n$ with radius $|r|$. If $r > 0$, the unit normal ξ of $S(p, r)$ is outward; if $r < 0$, the unit normal ξ of $S(p, r)$ is inward. If $r = 0$, then $S(p, r) \subset U\mathbb{R}^n$ consists of all unit tangent vector at p . We call $S(p, 0)$ the point sphere at $p \in \mathbb{R}^n$. An oriented hyperplane in $U\mathbb{R}^n$ with constant unit normal $\xi \in S^{n-1}$ and constant $\lambda \in \mathbb{R}$ is the $(n-1)$ -dimensional submanifold in $U\mathbb{R}^n$ given by

$$(2.3) \quad P(\xi, \lambda) = \{(x, \xi) \in U\mathbb{R}^n \mid x \cdot \xi = \lambda\}.$$

Geometrically, it is the hyperplane $\{x \in \mathbb{R}^n \mid x \cdot \xi = \lambda\}$ in \mathbb{R}^n with the unit normal ξ .

We denote by Σ the set of all oriented spheres and oriented hyperplanes in $U\mathbb{R}^n$. If $\gamma_1, \gamma_2 \in \Sigma$ satisfies $\gamma_1 = \gamma_2$, or they intersect in a single point $(x, \xi) \in U\mathbb{R}^n$, we say that γ_1 and γ_2 are oriented contact. Geometrically, γ_1, γ_2 are oriented contact at (x, ξ) if and only if they are spheres in \mathbb{R}^n which touch in x with the same unit normal ξ . We note that any point $(x, \xi) \in U\mathbb{R}^n$ determines uniquely a pencil of oriented spheres contact at $x \in \mathbb{R}^n$ with the common unit normal ξ . We note also that there is a unique point sphere $S(x, 0)$ and a unique hyperplane $P(\xi, x \cdot \xi)$ in this pencil.

Let \mathbb{R}_2^{n+3} be the space \mathbb{R}^{n+3} , equipped with the inner product

$$(2.4) \quad \langle X, Y \rangle = -X_1Y_1 + X_2Y_2 + \cdots + X_{n+2}Y_{n+2} - X_{n+3}Y_{n+3}.$$

Let C^{n+2} be the light-cone in \mathbb{R}^{n+3} given by

$$(2.5) \quad C^{n+2} = \{X \in \mathbb{R}_2^{n+3} \mid \langle X, X \rangle = 0\}.$$

We denote by \mathbb{Q}^{n+1} the quadric in the real projective space $R\mathbb{P}^{n+2}$, defined by

$$(2.6) \quad \mathbb{Q}^{n+1} = \{[X] \in R\mathbb{P}^{n+1} \mid \langle X, X \rangle = 0\}.$$

Then we can assign an oriented sphere $S(p, r) \in \Sigma$ to a point $[\gamma] \in \mathbb{Q}^{n+1}$ by

$$(2.7) \quad S(p, r) \leftrightarrow [\gamma], \quad \gamma = \left(\frac{1}{2}(1 + |p|^2 - r^2), \frac{1}{2}(1 - |p|^2 + r^2), p, -r\right)$$

and assign an oriented hyperplane $P(\xi, \lambda) \in \Sigma$ to a point in $[\gamma] \in \mathbb{Q}^{n+1}$ by

$$(2.8) \quad P(\xi, \lambda) \leftrightarrow [\gamma], \quad \gamma = (\lambda, -\lambda, \xi, 1).$$

We call $[\gamma] \in \mathbb{Q}^{n+1}$ the coordinate of the oriented sphere $S(p, r)$ or $P(\xi, \lambda)$.

For any $\gamma \in \Sigma$ we will denote by $[\gamma] \in \mathbb{Q}^{n+1}$ its coordinate given in (2.7) and (2.8). It is easy to verify that the corresponding $\gamma \in \Sigma \rightarrow [\gamma] \in \mathbb{Q}^{n+1}$ defines a bijection from Σ to $\mathbb{Q}^{n+1} \setminus \{[\wp]\}$, where

$$(2.9) \quad \wp = (1, -1, \mathbf{0}, 0), \quad \mathbf{0} \in \mathbb{R}^n.$$

Geometrically, the point

$$[\wp] = \lim_{|p| \rightarrow \infty} \frac{1}{|p|^2} \left[\left(\frac{1}{2}(1 + |p|^2), \frac{1}{2}(1 - |p|^2), p, 0 \right) \right]$$

in \mathbb{Q}^{n+1} is the coordinate of the point sphere at ∞ of \mathbb{R}^n . Using (2.7) and (2.8) we can easily verify that $\gamma_1, \gamma_2 \in \Sigma$ are oriented contact if and only if their sphere coordinates $[\gamma_1]$ and $[\gamma_2]$ satisfy

$$(2.10) \quad \langle \gamma_1, \gamma_2 \rangle = 0.$$

From (2.7), (2.8) and (2.9) we know that $[\gamma] \in \mathbb{Q}^{n+1} \setminus \{[\wp]\}$ is an oriented sphere $S(p, r)$ in $U\mathbb{R}^n$ if and only if $\langle \gamma, \wp \rangle \neq 0$ and that $[\gamma] \in \mathbb{Q}^{n+1} \setminus \{[\wp]\}$ is a hyperplane $P(\xi, \lambda)$ in $U\mathbb{R}^n$ if and only if $\langle \gamma, \wp \rangle = 0$.

A point $(x, \xi) \in U\mathbb{R}^n$ determines a unique pencil of oriented spheres contact at $x \in \mathbb{R}^n$ with the common unit normal ξ , and the point sphere $\gamma_1 = S(x, 0)$ and the oriented hyperplane $\gamma_2 = P(\xi, x \cdot \xi)$ in the pencil have coordinate $[\gamma_1]$ and $[\gamma_2]$, where

$$(2.11) \quad \gamma_1 = \left(\frac{1}{2}(1 + |x|^2), \frac{1}{2}(1 - |x|^2), x, 0\right), \quad \gamma_2 = (x \cdot \xi, -x \cdot \xi, \xi, 1).$$

Then any oriented sphere $[\gamma]$ in the pencil can be written as

$$(2.12) \quad [\gamma] = [\lambda\gamma_1 + \mu\gamma_2] \in \mathbb{Q}^{n+1} \setminus \{[\wp]\}$$

for some $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{0\}$. Thus a point $(x, \xi) \in U\mathbb{R}^n$ determines a unique projective line

$$\{[\lambda\gamma_1 + \mu\gamma_2] \mid (\lambda, \mu) \in \mathbb{R}^2 \setminus \{0\}\}$$

lying in $\mathbb{Q}^{n+1} \setminus \{[\wp]\}$.

Let Λ^{2n-1} be the set consisting of all projective lines lying in $\mathbb{Q}^{n+1} \setminus \{[\wp]\}$. Then the mapping $L : U\mathbb{R}^n \rightarrow \Lambda^{2n-1}$ defined by

$$(2.13) \quad L((x, \xi)) = \{[\lambda\gamma_1 + \mu\gamma_2] \in \mathbb{Q}^{n+1} \setminus \{[\wp]\} \mid (\lambda, \mu) \in \mathbb{R}^2 \setminus \{0\}\}$$

is a diffeomorphism, called Lie diffeomorphism.

§ 3. Laguerre transformation group on $U\mathbb{R}^n$

Let \mathbb{G} be the subgroup of Lorentzian group $O(n+1, 2)$ on \mathbb{R}_2^{n+3} given by

$$(3.1) \quad L\mathbb{G} = \{T \in O(n+1, 2) \mid \wp T = \wp\},$$

where \wp is the light-like vector in \mathbb{R}_2^{n+3} defined by (2.9). Then any $T \in L\mathbb{G}$ induces a transformation on \mathbb{Q}^{n+1} defined by

$$(3.2) \quad T([X]) = [XT], \quad X \in \mathbb{Q}^{n+1}.$$

We call both $T \in L\mathbb{G}$ and $T : \mathbb{Q}^{n+1} \rightarrow \mathbb{Q}^{n+1}$ Laguerre transformation.

Let $\gamma_1, \gamma_2 \in \Sigma$ be two different oriented contact spheres or hyperplanes. Then $[\gamma_1]$ and $[\gamma_2]$ define a projective line lying in $\mathbb{Q}^{n+1} \setminus \{[\wp]\}$ by

$$\text{span}\{[\gamma_1], [\gamma_2]\} = \{[\lambda\gamma_1 + \mu\gamma_2] \mid (\lambda, \mu) \in \mathbb{R}^2 \setminus \{0\}\} \in \Lambda^{2n-1}.$$

Then any $T \in L\mathbb{G}$ defines a transformation $T : \Lambda^{2n-1} \rightarrow \Lambda^{2n-1}$ by

$$T(\text{span}\{[\gamma_1], [\gamma_2]\}) = \text{span}\{[\gamma_1 T], [\gamma_2 T]\}.$$

Let $L : U\mathbb{R}^n \rightarrow \Lambda^{2n-1}$ be the Lie diffeomorphism. Then any $T \in L\mathbb{G}$ induces a transformation

$$\sigma = L^{-1} \circ T \circ L : U\mathbb{R}^n \rightarrow U\mathbb{R}^n,$$

called a Laguerre transformation on $U\mathbb{R}^n$. Thus the Laguerre transformation group on $U\mathbb{R}^n$ is given by

$$L\mathbb{G} = \{\sigma : U\mathbb{R}^n \rightarrow U\mathbb{R}^n \mid \sigma = L^{-1} \circ T \circ L, T \in O(n+1, 2), \wp T = \wp\}.$$

The dimension of $L\mathbb{G}$ is $(n+2)(n+1)/2$.

Let $T \in L\mathbb{G}$ be a Laguerre transformation. Then we have $\wp T = \wp$ and $T : \mathbb{Q}^{n+1} \setminus \{[\wp]\} \rightarrow \mathbb{Q}^{n+1} \setminus \{[\wp]\}$. Since any oriented sphere $\gamma \in U\mathbb{R}^n$ determines a point $[\gamma] \in \mathbb{Q}^{n+1} \setminus \{[\wp]\}$ such that $\langle \gamma, \wp \rangle \neq 0$, then $\langle \gamma T, \wp \rangle = \langle \gamma T, \wp T \rangle = \langle \gamma, \wp \rangle \neq 0$, we know that $T([\gamma])$ is also an oriented sphere. Since any oriented hyperplane $\gamma \in U\mathbb{R}^n$ determines a point $[\gamma] \in \mathbb{Q}^{n+1} \setminus \{[\wp]\}$ such that $\langle \gamma, \wp \rangle = 0$, then $\langle \gamma T, \wp \rangle = \langle \gamma T, \wp T \rangle = \langle \gamma, \wp \rangle = 0$, we know that $T([\gamma])$ is also an oriented hyperplane. Thus any Laguerre transformation $\sigma : U\mathbb{R}^n \rightarrow U\mathbb{R}^n$ takes oriented spheres to oriented spheres, takes oriented hyperplanes to oriented hyperplanes.

Example 3.1. Any isometry in \mathbb{R}^n given by

$$\sigma(x) = xA + a, \quad A \in O(n), a \in \mathbb{R}^n$$

induces an isometry transformation $\sigma : U\mathbb{R}^n \rightarrow U\mathbb{R}^n$ defined by

$$(3.3) \quad \sigma((x, \xi)) = (xA + a, \xi A).$$

It is easy to check that σ is a Laguerre transformation on $U\mathbb{R}^n$ and that

$$(3.4) \quad T(\sigma) = L \circ \sigma \circ L^{-1} = \begin{pmatrix} 1 + |a|^2/2 & -|a|^2/2 & a & 0 \\ |a|^2/2 & 1 - |a|^2/2 & a & 0 \\ Aa' & -Aa' & A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in L\mathbb{G},$$

where a' is the transport of the vector $a \in \mathbb{R}^n$.

Example 3.2. The 1-parametric parabolic transformations (or parallel transformations) defined by

$$(3.5) \quad \phi_t(x, \xi) = (x + t\xi, \xi), \quad t \in \mathbb{R}.$$

are Laguerre transformations in $U\mathbb{R}^n$. It is easy to check that

$$(3.6) \quad T(\phi_t) = L \circ \phi_t \circ L^{-1} = \begin{pmatrix} 1 - t^2/2 & t^2/2 & 0 & -t \\ -t^2/2 & 1 + t^2/2 & 0 & -t \\ 0 & 0 & I_n & 0 \\ t & -t & 0 & 1 \end{pmatrix} \in L\mathbb{G}.$$

Since $\phi_s \circ \phi_t = \phi_{s+t}$, we call ϕ_t a parabolic flow in $U\mathbb{R}^n$.

Example 3.3. The third example of Laguerre transformations in $U\mathbb{R}^n$ is the following 1-parametric hyperbolic transformations. For any $(x, \xi) \in U\mathbb{R}^n$ we write

$$x = (x_0, x_1) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad \xi = (\xi_0, \xi_1) \in \mathbb{R}^{n-1} \times \mathbb{R},$$

then a hyperbolic transformation

$$\psi_t(x, \xi) = (x(t), \xi(t)) \in U\mathbb{R}^n, \quad t \in \mathbb{R}$$

is defined by

$$(3.7) \quad x(t) = \left(x_0 - \frac{\sinh tx_1}{\sinh t\xi_1 + \cosh t} \xi_0, \frac{x_1}{\sinh t\xi_1 + \cosh t} \right);$$

$$(3.8) \quad \xi(t) = \left(\frac{1}{\sinh t\xi_1 + \cosh t} \xi_0, \frac{\cosh t\xi_1 + \sinh t}{\sinh t\xi_1 + \cosh t} \right).$$

It is easy to check that

$$(3.9) \quad T(\psi_t) = L \circ \psi_t \circ L^{-1} = \begin{pmatrix} I_{n+1} & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} \in L\mathbb{G}.$$

Since $\psi_s \circ \psi_t = \psi_{s+t}$, we call ψ_t a hyperbolic flow in $U\mathbb{R}^n$.

Let $\{e_1, e_2, \dots, e_{n+3}\}$ be the standard basis for \mathbb{R}_2^{n+3} , $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. For any $T \in L\mathbb{G}$ we have

$$\wp T = \wp, \quad \langle e_i T, \wp \rangle = \langle e_i T, \wp T \rangle = \langle e_i, \wp \rangle, \quad 1 \leq i \leq n+3.$$

Using these information and the fact that $T \in O(n+1, 2)$ we can write

$$(3.10) \quad T = \begin{pmatrix} 1 + |a|^2/2 - \rho^2/2 & -|a|^2/2 + \rho^2/2 & a & \rho \\ |a|^2/2 - \rho^2/2 & 1 - |a|^2/2 + \rho^2/2 & a & \rho \\ Aa' - \rho u & -Aa' + \rho u & A & u \\ va' - \rho w & -va' + \rho w & v & w \end{pmatrix}$$

for some

$$(3.11) \quad \begin{pmatrix} A & u \\ v & w \end{pmatrix} \in O(n, 1), \quad (a, \rho) \in \mathbb{R}^{n+1}, w \in \mathbb{R}.$$

It is easy to check that

$$(3.12) \quad T \rightarrow \begin{pmatrix} A & u & 0 \\ v & w & 0 \\ a & \rho & 1 \end{pmatrix}$$

is a isomorphism from $L\mathbb{G}$ to the Lorentzian transformation group in \mathbb{R}_1^{n+1} .

Now let $\gamma_1 = S(p, r)$, $\gamma_2 = S(p^*, r^*)$ be oriented spheres in \mathbb{R}^n . Let T be a Laguerre transformation given by (3.10). Since

$$\gamma_1 = \left(\frac{1}{2}(1 + |p|^2 - r^2), \frac{1}{2}(1 - |p|^2 + r^2), p, -r\right),$$

$$\gamma_2 = \left(\frac{1}{2}(1 + |p^*|^2 - r^{*2}), \frac{1}{2}(1 - |p^*|^2 + r^{*2}), p^*, -r^*\right),$$

then the oriented spheres $\gamma_1 T = S(\tilde{p}, \tilde{r})$ and $\gamma_2 T = S(\tilde{p}^*, \tilde{r}^*)$ are given by

$$(\tilde{p}, -\tilde{r}) = (pA - rv + a, pu + rw + \rho), \quad (\tilde{p}^*, -\tilde{r}^*) = (p^*A - r^*v + a, p^*u + r^*w + \rho).$$

Thus we have

$$(3.13) \quad (\tilde{p}^* - \tilde{p}, -\tilde{r}^* + \tilde{r}) = (p^* - p, -r^* + r) \begin{pmatrix} A & u \\ v & w \end{pmatrix}.$$

It follows that

$$(3.14) \quad F = |p^* - p|^2 - (r^* - r)^2$$

is a Laguerre invariant. Geometrically, if one sphere is not contained in another, then F is exactly the square length of the common tangent segment of the two spheres $S(p, r)$ and $S(p^*, r^*)$.

Theorem 3.1 *For any $T \in O(n+1, 2)$ with $\wp T = T$ there exist two isometries σ_1, σ_2 on $U\mathbb{R}^n$ and constants $s, t \in \mathbb{R}, \varepsilon = \pm 1$ such that*

$$(3.15) \quad T = \varepsilon T(\sigma_2)T(\psi_t)T(\phi_s)T(\sigma_1).$$

Proof. For any $T \in O(n+1, 2)$ with $\wp T = T$ we can write T as in (3.10). From (3.11) we get $w^2 = 1 + |v|^2$. By changing T to $-T$, if necessary, we may assume that $w > 0$. Then we have

$$e_{n+3}T = ((va' - \rho w), -(va' - \rho w), v, w), \quad w = \sqrt{|v|^2 + 1}.$$

We can find $s, t \in \mathbb{R}$ and $A_1 \in O(n)$

$$s = w^{-1}(va' - \rho w), \quad w = \cosh t, \quad vA_1 = (0, \dots, 0, \sinh t).$$

We denote by σ_1^{-1} the isometry $\sigma_1^{-1}((x, \xi)) = (xA_1, \xi A_1)$ on $U\mathbb{R}^n$, then by (3.4), (3.6) and (3.9) we have

$$e_{n+3}TT(\sigma_1^{-1})T(\phi_{-s})T(\psi_{-t}) = e_{n+3}.$$

We define

$$T^* = TT(\sigma_1^{-1})T(\phi_{-s})T(\psi_{-t}).$$

Since T^* satisfies

$$T^* \in O(n+1, 2), \quad \wp T^* = \wp, \quad e_{n+3} T^* = e_{n+3},$$

T^* takes the form (3.10) with $e_{n+3} = (0, \dots, 0, 1)$ as its last line. Thus T^* takes the form (3.4) for some isometry σ_2 . Thus we get (3.15) and complete the proof of Theorem 2.1.

Corollary *Any Laguerre transformation in $U\mathbb{R}^n$ is generated by the isometries, the parallel transformations and the hyperbolic transformations.*

§ 4. Laguerre space forms and Laguerre embeddings

In Moebius geometry we have three standard spaces S^n , \mathbb{R}^n and \mathbb{H}^n and conformal embeddings $\mathbb{R}^n \rightarrow S^n$ and $\mathbb{H}^n \rightarrow S^n$. Similarly, we introduce in this section three Laguerre space forms $U\mathbb{R}^n$, $U\mathbb{R}_1^n$ and $U\mathbb{R}_0^n$ and the Laguerre embeddings $\sigma : U\mathbb{R}_1^n \rightarrow U\mathbb{R}^n$ and $\tau : U\mathbb{R}_0^n \rightarrow U\mathbb{R}^n$.

Let \mathbb{R}_1^n be the Lorentzian space with inner product

$$(4.1) \quad \langle X, Y \rangle = X_1 Y_1 + \dots + X_{n-1} Y_{n-1} - X_n Y_n.$$

Let $U\mathbb{R}_1^n$ be the unit bundle of \mathbb{R}_1^n defined by

$$(4.2) \quad U\mathbb{R}_1^n = \{(x, \xi) \mid x \in \mathbb{R}_1^n, \langle x, \xi \rangle = -1\}.$$

An oriented sphere (hyperboloid) $H(p, r)$ centered at p in \mathbb{R}_1^n with radius r can be embedded in $U\mathbb{R}_1^n$ as the $(n-1)$ -dimensional submanifold

$$(4.3) \quad H(p, r) = \{(x, \xi) \in U\mathbb{R}_1^n \mid x - p = r\xi\}.$$

Here r is a real number. If $r = 0$, then $H(p, r)$ consists all unit time-like vectors at p , called "point sphere" at p . We assign $\gamma = H(p, r)$ a vector $[\gamma] \in \mathbb{Q}^{n+1}$ by

$$(4.4) \quad \gamma = \left(\frac{1}{2}(1 + \langle p, p \rangle + r^2), \frac{1}{2}(1 - \langle p, p \rangle - r^2), -r, p\right).$$

An oriented space-like hyperplane $P(\xi, \lambda)$ in \mathbb{R}_1^n with unit normal ξ can be embedded in $U\mathbb{R}_1^n$ as the $(n-1)$ -dimensional submanifold

$$(4.5) \quad P(\xi, \lambda) = \{(x, \xi) \in U\mathbb{R}_1^n \mid \langle x, \xi \rangle = \lambda\}.$$

We assign $\gamma = P(\xi, \lambda)$ the vector $[\gamma] \in \mathbb{Q}^{n+1}$ by

$$(4.6) \quad \gamma = (\lambda, -\lambda, 1, \xi).$$

Two oriented spheres (or hyperplanes) γ_1 and γ_2 are oriented contact in \mathbb{R}_1^n if and only if their corresponding vectors $\gamma_1, \gamma_2 \in \mathbb{Q}^{n+1}$ satisfy $\langle \gamma_1, \gamma_2 \rangle = 0$. Any point

$(x, \xi) \in U\mathbb{R}_1^n$ determines a pencil of spheres (hyperplanes) in \mathbb{R}_1^n oriented contacted at x with common normal vector ξ , which corresponds to a projective line by the Lie diffeomorphism $L_1 : U\mathbb{R}_1^n \rightarrow \Lambda^{2n-1}$ given by

$$(4.7) \quad L_1(x, \xi) = \{[\lambda\gamma_1 + \mu\gamma_2] \mid (\lambda, \mu) \in \mathbb{R}^2 \setminus \{0\}\},$$

where

$$(4.8) \quad \gamma_1 = \left(\frac{1}{2}(1 + \langle x, x \rangle), \frac{1}{2}(1 - \langle x, x \rangle), 0, x\right),$$

$$(4.9) \quad \gamma_2 = (\langle x, \xi \rangle, -\langle x, \xi \rangle, 1, \xi).$$

Let $L : U\mathbb{R}^n \rightarrow \Lambda^{2n-1}$ be the Lie diffeomorphism defined by (2.13). It is easy to check that $\sigma = L^{-1} \circ L_1 : U\mathbb{R}_1^n \rightarrow U\mathbb{R}^n$ given by

$$(4.10) \quad \sigma(x, \xi) = (x', \xi') \in U\mathbb{R}^n;$$

where $(x, \xi) \in U\mathbb{R}_1^n$ with $x = (x_0, x_1) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $\xi = (\xi_0, \xi_1) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and

$$(4.11) \quad x' = \left(-\frac{x_1}{\xi_1}, x_0 - \frac{x_1}{\xi_1}\xi_0\right), \quad \xi' = \left(\frac{1}{\xi_1}, \frac{\xi_0}{\xi_1}\right).$$

It is straightforward to verify that σ takes the hyperplane $P(\xi, \lambda)$ in $U\mathbb{R}_1^n$ to the hyperplane $P(\xi', \lambda/\xi_1)$ in $U\mathbb{R}^n$, takes the oriented sphere $H(p, r)$ in $U\mathbb{R}_1^n$ into the oriented sphere $S(p', r')$ in $U\mathbb{R}^n$, where $p = (p_0, p_1)$, $p' = (-r, p_0)$ and $r' = -p_1$. Thus $\sigma : U\mathbb{R}_1^n \rightarrow U\mathbb{R}^n$ is a Laguerre embedding.

Let \mathbb{R}_1^{n+1} be the Lorentzian space with inner product

$$(4.12) \quad \langle X, Y \rangle = X_1Y_1 + \cdots + X_nY_n - X_{n+1}Y_{n+1}.$$

Let $\nu = (1, \mathbf{0}, 1)$ be the light-like vector in \mathbb{R}_1^{n+1} with $\mathbf{0} \in \mathbb{R}^{n-1}$. Let \mathbb{R}_0^n be the degenerate hyperplane in \mathbb{R}_1^{n+1} defined by

$$(4.13) \quad \mathbb{R}_0^n = \{X \in \mathbb{R}_1^{n+1} \mid \langle X, \nu \rangle = 0\}.$$

We define

$$(4.14) \quad U\mathbb{R}_0^n = \{(x, \xi) \in \mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} \mid \langle x, \nu \rangle = 0, \langle \xi, \xi \rangle = 0, \langle \xi, \nu \rangle = 1\}.$$

An oriented sphere $C(p)$ in \mathbb{R}_0^n with $p \in \mathbb{R}_1^{n+1}$ is the $(n-1)$ -submanifold in $U\mathbb{R}_0^n$ given by

$$(4.15) \quad C(p) = \{(x, \xi) \in U\mathbb{R}_0^n \mid x - p = -\langle p, \nu \rangle \xi\}.$$

Geometrically, $C(p)$ ($\langle p, \nu \rangle \neq 0$) is a paraboloid in \mathbb{R}_0^n as the intersection of the light-cone $\langle X - p, X - p \rangle = 0$ in \mathbb{R}_1^{n+1} with the degenerate hyperplane $\langle X, \nu \rangle = 0$.

The paraboloid $C(p)$ is centered at $p^* = p + (r, \mathbf{0}, 0) \in \mathbb{R}_0^n$ ($r = -\langle p, \nu \rangle$) with the symmetric axe $\ell = \{p^* + t\nu \mid t \in \mathbb{R}\}$. If $r = 0$, then $C(p)$ consists of all $(p, \xi) \in U\mathbb{R}_0^n$ with ξ lying on the paraboloid $\{\xi \in \mathbb{R}_1^{n+1} \mid \langle \xi, \xi \rangle = 0, \langle \xi, \nu \rangle = 1\}$ in \mathbb{R}_1^{n+1} . We assign $\gamma = C(p)$ a vector $[\gamma] \in \mathbb{Q}^{n+1}$ by

$$(4.16) \quad \gamma = \left(\frac{1}{2}(1 + \langle p, p \rangle), \frac{1}{2}(1 - \langle p, p \rangle), p\right).$$

An oriented space-like hyperplane $P(\xi, \lambda)$ in \mathbb{R}_0^n with unit normal ξ can be embedded in $U\mathbb{R}_0^n$ as the $(n-1)$ -submanifold

$$(4.17) \quad P(\xi, \lambda) = \{(x, \xi) \in U\mathbb{R}_0^n \mid \langle x, \xi \rangle = \lambda\}.$$

We assign $\gamma = P(\xi, \lambda)$ the vector $[\gamma] \in \mathbb{Q}^{n+1}$ by

$$(4.18) \quad \gamma = (\lambda, -\lambda, \xi).$$

Two oriented spheres (or hyperplanes) γ_1 and γ_2 are oriented contact in \mathbb{R}_0^n if and only if their corresponding vectors $\gamma_1, \gamma_2 \in \mathbb{Q}^{n+1}$ satisfy $\langle \gamma_1, \gamma_2 \rangle = 0$. Any point $(x, \xi) \in U\mathbb{R}_0^n$ determines a pencil of spheres (hyperplanes) in \mathbb{R}_0^n oriented contacted at x with common normal vector ξ , which corresponds to a projective line by the Lie diffeomorphism $L_0 : U\mathbb{R}_0^n \rightarrow \Lambda^{2n-1}$

$$(4.19) \quad L_0(x, \xi) = \{[\lambda\gamma_1 + \mu\gamma_2] \mid (\lambda, \mu) \in \mathbb{R}^2 \setminus \{0\}\},$$

where

$$(4.20) \quad \gamma_1 = \left(\frac{1}{2}(1 + \langle x, x \rangle), \frac{1}{2}(1 - \langle x, x \rangle), x\right),$$

$$(4.21) \quad \gamma_2 = (\langle x, \xi \rangle, -\langle x, \xi \rangle, \xi).$$

Let $L : U\mathbb{R}^n \rightarrow \Lambda^{2n-1}$ be the Lie diffeomorphism defined by (2.13). It is easy to check that $\tau = L^{-1} \circ L_0 : U\mathbb{R}_1^n \rightarrow U\mathbb{R}^n$ given by

$$(4.22) \quad \tau(x, \xi) = (x', \xi') \in U\mathbb{R}^n;$$

where $x = (x_1, x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, $\xi = (\xi_1 + 1, \xi_0, \xi_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ and

$$(4.23) \quad x' = \left(-\frac{x_1}{\xi_1}, x_0 - \frac{x_1}{\xi_1}\xi_0\right), \quad \xi' = \left(1 + \frac{1}{\xi_1}, \frac{\xi_0}{\xi_1}\right).$$

It is straightforward to verify that τ takes the hyperplane $P(\xi, \lambda)$ in $U\mathbb{R}_0^n$ to the hyperplane $P(\xi', \lambda/\xi_1)$ in $U\mathbb{R}^n$, takes the oriented sphere $C(p)$ in $U\mathbb{R}_0^n$ into the oriented sphere $S(p', r')$ in $U\mathbb{R}^n$, where $p = (p_1 - r, p_0, p_1)$, $p' = (p_1 - r, p_0)$, $r = -\langle p, \nu \rangle$ and $r' = -p_1$. Thus $\tau : U\mathbb{R}_0^n \rightarrow U\mathbb{R}^n$ is a Laguerre embedding.

§ 5. Laguerre hypersurfaces in $U\mathbb{R}^n$

Let $x : U\mathbb{R}^n \rightarrow \mathbb{R}^n$, $\xi : U\mathbb{R}^n \rightarrow S^{n-1} \subset \mathbb{R}^n$ be the standard projections $(x, \xi) \rightarrow x$ and $(x, \xi) \rightarrow \xi$, respectively. Then there is a standard contact form ω in $U\mathbb{R}^n$ defined by

$$(5.1) \quad \omega = dx \cdot \xi.$$

It is easy to verify that $\omega \wedge d\omega^{n-1} \neq 0$, which is (up to a non-zero constant) the volume form for the embedding of $U\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$ in \mathbb{R}^{2n} .

Let $(x, \xi) : U\mathbb{R}^n \rightarrow \mathbb{R}^n \times S^{n-1} \subset \mathbb{R}^{2n}$ be the standard embedding. We define $\gamma_1, \gamma_2 : U\mathbb{R}^n \rightarrow \mathbb{R}_2^{n+3}$ by (2.11). Let $T \in L\mathbb{G}$ be a Laguerre transformation and

$$(\tilde{x}, \tilde{\xi}) = \phi((x, \xi)), \quad \phi = L^{-1} \circ T \circ L : U\mathbb{R}^n \rightarrow U\mathbb{R}^n.$$

We denote by a, b the last coordinate of $\gamma_1 T$ and $\gamma_2 T$, respectively. Then by (2.11) and (3.10) we can write

$$(5.2) \quad \tilde{\gamma}_1 = \left(\frac{1}{2}(1 + |\tilde{x}|^2), \frac{1}{2}(1 - |\tilde{x}|^2), \tilde{x}, 0 \right) = \gamma_1 T - \frac{a}{b} \gamma_2 T,$$

$$(5.3) \quad \tilde{\gamma}_2 = (\tilde{x} \cdot \tilde{\xi}, -\tilde{x} \cdot \tilde{\xi}, \tilde{\xi}, 1) = \frac{1}{b} \gamma_2 T.$$

It follows that

$$(5.4) \quad d\tilde{x} \cdot \tilde{\xi} = \langle d\tilde{\gamma}_1, \tilde{\gamma}_2 \rangle = \langle d(\gamma_1 T - \frac{a}{b} \gamma_2 T), \frac{1}{b} \gamma_2 T \rangle = \frac{1}{b} \langle d\gamma_1, \gamma_2 \rangle = \frac{1}{b} dx \cdot \xi.$$

$$(5.5) \quad d\tilde{\xi} \cdot d\tilde{\xi} = \langle d\tilde{\gamma}_2, d\tilde{\gamma}_2 \rangle = \frac{1}{b^2} \langle d\gamma_2, d\gamma_2 \rangle = \frac{1}{b^2} d\xi \cdot d\xi.$$

We call $f = (x, \xi) : M^{n-1} \rightarrow U\mathbb{R}^n$ a Laguerre hypersurface, if $\xi : M^{n-1} \rightarrow \mathbb{R}^n$ is a immersion and $f^* \omega = dx \cdot \xi = 0$. It follows from (5.4) and (5.5) that any Laguerre transformation takes Laguerre hypersurfaces in $U\mathbb{R}^n$ to Laguerre hypersurfaces in $U\mathbb{R}^n$. By (2.2) and (2.3) we know that oriented spheres and hyperplanes are simplest Laguerre hypersurfaces in $U\mathbb{R}^n$.

Let $x : M^{n-1} \rightarrow \mathbb{R}^n$ be an oriented hypersurface in \mathbb{R}^n with non-zero principal curvatures. Then the unit normal $\xi : M \rightarrow \mathbb{R}^n$ is a immersion. Thus x induces uniquely a Laguerre hypersurface $f = (x, \xi) : M^{n-1} \rightarrow U\mathbb{R}^n$. We note that for a Laguerre hypersurface $f = (x, \xi) : M^{n-1} \rightarrow U\mathbb{R}^n$, $x : M^{n-1} \rightarrow \mathbb{R}^n$ may not be an immersion. By a theorem of U. Pinkall [9] we know that the parallel transformation $f_t = (x + t\xi, \xi)$ of f is an immersion at any given point $p \in M^{n-1}$ for almost all $t \in \mathbb{R}$. In this sense we may assume that $x : M^{n-1} \rightarrow \mathbb{R}^n$ is an immersion.

Let $x, \tilde{x} : M^{n-1} \rightarrow \mathbb{R}^n$ be two oriented hypersurfaces with non-zero principal curvatures. We say x, \tilde{x} are Laguerre equivalent, if the corresponding Laguerre hypersurfaces $f = (x, \xi), \tilde{f} = (\tilde{x}, \tilde{\xi}) : M^{n-1} \rightarrow U\mathbb{R}^n$ are differ only by a Laguerre transformation $\phi : U\mathbb{R}^n \rightarrow U\mathbb{R}^n$, i.e., $\tilde{f} = \phi \circ f$. In Laguerre differential geometry we study properties of Laguerre hypersurfaces in $U\mathbb{R}^n$ which are invariant under the Laguerre transformation group in $U\mathbb{R}^n$.

Let $x : M \rightarrow \mathbb{R}^n$ be oriented hypersurface with unit normal ξ . We define

$$(5.6) \quad [y] : M \rightarrow Q^{n+1}, \quad y = (x \cdot \xi, -x \cdot \xi, \xi, 1).$$

Theorem 5.1 *Let $x, x^* : M \rightarrow \mathbb{R}^n$ be two oriented hypersurfaces with non-zero principal curvatures. Then x and x^* are Laguerre equivalent if and only if there exists $T \in L\mathbb{G}$ such that $[y^*] = [yT]$.*

Proof. Let ξ and ξ^* be the unit normal of x and x^* , respectively. If there is a Laguerre transformation $\phi = L^{-1} \circ T \circ L \in L\mathbb{G}$ such that $(x^*, \xi^*) = \phi \circ (x, \xi)$, then by (5.3) we obtain $[y^*] = [yT]$. Conversely, if $[y^*] = [yT]$ for some $T \in L\mathbb{G}$, we define $(\tilde{x}, \tilde{\xi}) = \phi \circ (x, \xi)$ with $\phi = L^{-1} \circ T \circ L$. Then by (5.3) we have $[\tilde{y}] = [yT] = [y^*]$. It follows that

$$(5.7) \quad (\tilde{x} \cdot \tilde{\xi}, -\tilde{x} \cdot \tilde{\xi}, \tilde{\xi}, 1) = (x^* \cdot \xi^*, -x^* \cdot \xi^*, \xi^*, 1).$$

Let $\{e_i\}$ be a local basis for TM . Since $\xi^* : M \rightarrow \mathbb{R}^n$ is an immersion, we know that $\{e_1(\xi^*), \dots, e_{n-1}(\xi^*), \xi^*\}$ is a basis for \mathbb{R}^n . From (5.7) and the facts that

$$\xi^* = \tilde{\xi}, \quad (x^* - \tilde{x}) \cdot \xi^* = 0, \quad (x^* - \tilde{x}) \cdot d\xi^* = d((x^* - \tilde{x}) \cdot \xi^*) = 0,$$

we get $x^* = \tilde{x}$. Thus we have $(x^*, \xi^*) = \phi \circ (x, \xi)$, which implies that x and x^* are Laguerre equivalent. We complete the proof of Theorem 5.1.

Since by (5.6) we have $\langle dy, dy \rangle = d\xi \cdot d\xi$, which is exactly the third fundamental form of x . It follows from Theorem 5.1 that

Corollary *The conformal class of the third fundamental form of a hypersurface $x : M \rightarrow \mathbb{R}^n$ is a Laguerre invariant.*

Let $x : M \rightarrow \mathbb{R}^n$ be a oriented hypersurface with non-zero principal curvatures. Let $III = \langle dy, dy \rangle$ be the third fundamental form of x . For any orthonormal basis $\{E_1, E_2, \dots, E_{n-1}\}$ with respect to III we define

$$(5.8) \quad \mathbb{V} = \text{span}\{y, \Delta y, E_1(y), E_2(y), \dots, E_{n-1}(y)\},$$

where Δ is the Laplacian operator with respect to $III = \langle dy, dy \rangle$. Then we have

$$(5.9) \quad \langle y, E_i(y) \rangle = \langle \Delta y, E_i(y) \rangle = 0, \quad \langle y, \Delta y \rangle = -(n-1), \quad \langle E_i(y), E_j(y) \rangle = \delta_{ij}.$$

Thus at each point \mathbb{V} is a $(n + 1)$ -dimensional non-degenerate subspace in \mathbb{R}_2^{n+3} of type $(-, +, \dots, +)$. Let

$$(5.10) \quad \mathbb{R}_2^{n+3} = \mathbb{V} \oplus \mathbb{V}^\perp = \text{span}\{y, \Delta y, E_1(y), E_2(y), \dots, E_{n-1}(y)\} \oplus \mathbb{V}^\perp$$

be the orthogonal decomposition of \mathbb{R}_2^{n+3} . Then \mathbb{V}^\perp is a 2-dimensional non-degenerate subspace of \mathbb{R}_2^{n+3} of type $(-, +)$.

Let $\{e_1, e_2, \dots, e_{n-1}\}$ be the orthonormal basis for TM with respect to $dx \cdot dx$, consisting of unit principal vectors. We write the structure equation of $x : M \rightarrow \mathbb{R}^n$ by

$$(5.11) \quad e_j(e_i(x)) = \sum_k \Gamma_{ij}^k e_k(x) + k_i \delta_{ij} \xi; \quad e_i(\xi) = -k_i e_i(x), \quad 1 \leq i, j, k \leq n-1,$$

where $k_i \neq 0$ is the principal curvature corresponding to e_i . Let

$$(5.12) \quad r_i = \frac{1}{k_i}, \quad r = \frac{r_1 + r_2 + \dots + r_{n-1}}{n-1}$$

the curvature radius and mean curvature radius of x . Then the mean curvature sphere $S(x + r\xi, r)$ of x in \mathbb{R}^n has the sphere coordinate $[\eta]$, where

$$(5.13) \quad \eta = \left(\frac{1}{2}(1 + |x|^2), \frac{1}{2}(1 - |x|^2), x, 0\right) + r(x \cdot \xi, -x \cdot \xi, \xi, 1).$$

We define $E_i = r_i e_i, 1 \leq i \leq (n-1)$, then $\{E_1, E_2, \dots, E_{n-1}\}$ is an orthonormal basis for $III = \langle dy, dy \rangle$. From (5.11) we get

$$(5.14) \quad E_i(y) = -(x \cdot e_i(x), -x \cdot e_i(x), e_i(x), 0).$$

It follows from (5.6), (5.13) and (5.14) that

$$(5.15) \quad \langle y, \eta \rangle = 0, \quad \langle E_i(y), \eta \rangle = 0, \quad E_i(\eta) = (r - r_i)E_i(y) + E_i(r)y.$$

Moreover, from (5.15) we get

$$\langle \Delta y, \eta \rangle = \sum_i \langle E_i E_i(y), \eta \rangle = - \sum_i \langle E_i(y), E_i(\eta) \rangle = - \sum_i (r - r_i) = 0.$$

Thus we know that $\eta \in \mathbb{V}^\perp$. Let $\wp = (1, -1, \mathbf{0}, 0) \in \mathbb{R}_2^{n+3}$ be the vector defined by (2.9). Since $\langle y, \wp \rangle = 0$, by (5.13) we have

$$(5.16) \quad \mathbb{V}^\perp = \text{span}\{\eta, \wp\}, \quad \langle \eta, \eta \rangle = \langle \wp, \wp \rangle = 0, \quad \langle \eta, \wp \rangle = -1.$$

We call $\eta : M \rightarrow C^{n+2} \subset \mathbb{R}^{n+3}$ defined by (5.13) the Laguerre Gauss map of x .

It is clear that $\mathbb{V}, \mathbb{V}^\perp$ and η are Laguerre invariant: if x is Laguerre equivalent to \tilde{x} by $T \in L\mathbb{G}$, then we have

$$(5.17) \quad \tilde{\mathbb{V}} = \mathbb{V}T, \quad \tilde{\mathbb{V}}^\perp = \mathbb{V}^\perp T, \quad \tilde{\eta} = \eta T.$$

Now let $x, \tilde{x} : M \rightarrow \mathbb{R}^n$ are Laguerre equivalent by $T \in L\mathbb{G}$. Then by (5.3) and (5.17) we have

$$(5.18) \quad \tilde{y} = \frac{1}{b}yT, \quad \tilde{\eta} = \eta T$$

for function $b \neq 0$. It follows that

$$(5.19) \quad \langle d\tilde{y}, d\tilde{y} \rangle = \frac{1}{b^2} \langle dy, dy \rangle.$$

If $\{E_i\}$ is an orthonormal basis for $\langle dy, dy \rangle$, then $\{\tilde{E}_i = bE_i\}$ is an orthonormal basis for $\langle d\tilde{y}, d\tilde{y} \rangle$. From (5.18) and (5.19) we obtain

$$(5.20) \quad \sum_i \langle \tilde{E}_i(\tilde{\eta}), \tilde{E}_i(\tilde{\eta}) \rangle = b^2 \sum_i \langle E_i(\eta), E_i(\eta) \rangle.$$

It follows from (5.19) and (5.20) that

$$(5.21) \quad g = \left(\sum_i \langle E_i(\eta), E_i(\eta) \rangle \right) \langle dy, dy \rangle = \left(\sum_i \langle E_i(\eta), E_i(\eta) \rangle \right) III$$

is a Laguerre invariant. From the last equation of (5.15) we get

$$(5.22) \quad \sum_i \langle E_i(\eta), E_i(\eta) \rangle = \sum_i (r_i - r)^2.$$

Thus we know that

$$(5.23) \quad g = \left(\sum_i (r_i - r)^2 \right) III$$

is a Laguerre invariant metric at any non-umbilical point of x . We call g the Laguerre metric of x . The volume of g is given by

$$(5.24) \quad L(x) = Vol_g(x) = \int_M \frac{(\sum_i (r_i - r)^2)^{(n-1)/2}}{r_1 r_2 \cdots r_{n-1}} dM,$$

where dM is the volume form with respect to $dx \cdot dx$. We call critical hypersurfaces of the functional $L(x)$ Laguerre minimal hypersurfaces. In case $n = 3$, we get

$$(5.25) \quad L(x) = 2 \int_M \frac{H^2 - K}{K} dM,$$

which is the Laguerre functional given in Blaschke's book [1], the papers of E.Musso and L.Nicolodi [5] and B. Palmer [8] (up to a factor).

§ 6. Laguerre invariant system for hypersurfaces in \mathbb{R}^n

Let $x : M \rightarrow \mathbb{R}^n$ be a umbilical free hypersurface with non-zero principal curvatures. We define

$$(6.1) \quad Y = \rho(\xi, -x \cdot \xi, x \cdot \xi, 1), \quad \rho = \sqrt{\sum_i (r_i - r)^2} > 0.$$

If $x, \tilde{x} : M \rightarrow \mathbb{R}^n$ are Laguerre equivalent by $T \in L\mathbb{G}$, then by (5.18), (5.20) and (5.22) we obtain $\tilde{Y} = YT$. Thus

$$Y : M \rightarrow C^{n+2} \subset \mathbb{R}_2^{n+3}$$

is a Laguerre invariant. We call Y the Laguerre position vector of the hypersurface $x : M \rightarrow \mathbb{R}^n$. It follows immediately from Theorem 5.1 that

Theorem 6.1 *Let $x, \tilde{x} : M^{n-1} \rightarrow \mathbb{R}^n$ be two umbilical free oriented hypersurfaces with non-zero principal curvatures. Then x and \tilde{x} are Laguerre equivalent if and only if there exists $T \in \mathbb{G}$ such that $\tilde{Y} = YT$.*

Let Y the Laguerre position vector of a hypersurface $x : M \rightarrow \mathbb{R}^n$. Then the Laguerre metric g can be written as

$$(6.2) \quad g = \langle dY, dY \rangle.$$

We denote by Δ the Laplacian operator of g and define

$$(6.3) \quad N = \frac{1}{n-1} \Delta Y + \frac{1}{2(n-1)^2} \langle \Delta Y, \Delta Y \rangle Y.$$

Then we have

$$(6.4) \quad \langle Y, Y \rangle = \langle N, N \rangle = 0, \quad \langle Y, N \rangle = -1.$$

Let $\{E_1, E_2, \dots, E_{n-1}\}$ be an orthonormal basis for $g = \langle dY, dY \rangle$ with dual basis $\{\omega_1, \omega_2, \dots, \omega_{n-1}\}$. Then we have the following orthogonal decomposition

$$(6.5) \quad \mathbb{R}_2^{n+3} = \text{span}\{Y, N\} \oplus \text{span}\{E_1(Y), E_2(Y), \dots, E_{n-1}(Y)\} \oplus \{\eta, \wp\}.$$

We call $\{Y, N, E_1(Y), E_2(Y), \dots, E_{n-1}(Y), \eta, \wp\}$ a Laguerre moving frame in \mathbb{R}_2^{n+3} of x . By taking derivatives of this frame we obtain the following structure equations:

$$(6.6) \quad E_i(N) = \sum_j L_{ij} E_j(Y) + C_i \wp;$$

$$(6.7) \quad E_j(E_i(Y)) = L_{ij} Y + \delta_{ij} N + \sum_k \Gamma_{ij}^k E_k(Y) + B_{ij} \wp;$$

$$(6.8) \quad E_i(\eta) = -C_i Y + \sum_j B_{ij} E_j(Y).$$

From these equations we obtain the following basic Laguerre invariants:

- (i) The Laguerre metric $g = \langle dY, dY \rangle$;
- (ii) The Laguerre second fundamental form $\mathbb{B} = \sum_{ij} B_{ij} \omega_i \otimes \omega_j$;
- (iii) The symmetric 2-tensor $\mathbb{L} = \sum_{ij} L_{ij} \omega_i \otimes \omega_j$;
- (iv) The Laguerre form $C = \sum_i C_i \omega_i$.

By taking further derivatives of (6.6), (6.7) and (6.8) we get the following relations between these invariants:

$$(6.9) \quad L_{ij,k} = L_{ik,j};$$

$$(6.10) \quad C_{i,j} - C_{j,i} = \sum_k (B_{ik} L_{kj} - B_{jk} L_{ki});$$

$$(6.11) \quad B_{ij,k} - B_{ik,j} = C_j \delta_{ik} - C_k \delta_{ij};$$

$$(6.12) \quad R_{ijkl} = L_{jk} \delta_{il} + L_{il} \delta_{jk} - L_{ik} \delta_{jl} - L_{jl} \delta_{ik};$$

where $\{L_{ij,k}\}, \{C_{i,j}\}, \{B_{ij,k}\}$ are covariant derivatives with respect to the Laguerre metric g , and R_{ijkl} is the curvature tensor of g . Since $\{E'_i = \rho E_i\}$ is an orthonormal basis for the third fundamental form III , we get from (6.8) and (5.22) that

$$(6.13) \quad \sum_{ij} B_{ij}^2 = \rho^{-2} \sum_i \langle E'_i(\eta), E'_i(\eta) \rangle = \rho^{-2} \sum_i (r_i - r)^2 = 1.$$

Moreover, from (6.7) we have

$$\Delta Y = \left(\sum_i L_{ii} \right) Y + (n-1)N + \left(\sum_i B_{ii} \right) \wp.$$

It follows from (6.3) that

$$(6.14) \quad \sum_i B_{ii} = 0, \quad \sum_i L_{ii} = -\frac{1}{2(n-1)} \langle \Delta Y, \Delta Y \rangle.$$

From (6.11) we get

$$(6.15) \quad \sum_i B_{ij,i} = (n-2)C_j.$$

From (6.12) we get

$$(6.16) \quad R_{ik} = -(n-3)L_{ik} - \left(\sum_i L_{ii} \right) \delta_{ik};$$

$$(6.17) \quad R = -2(n-2) \sum_i L_{ii} = \frac{(n-2)}{(n-1)} < \Delta Y, \Delta Y > .$$

In case $n > 3$, we know from (6.15) and (6.16) that C_i and L_{ij} are completely determined by the Laguerre invariants $\{g, \mathbb{B}\}$, thus we get

Theorem 6.2 *Two umbilical free oriented hypersurfaces in \mathbb{R}^n ($n > 3$) with non-zero principal curvatures are Laguerre equivalent if and only if they have the same Laguerre metric g and Laguerre second fundamental form \mathbb{B} .*

Let $S : TM \rightarrow TM$ be the shape operator for x with principal radius r_i . By direct calculation for hypersurface $x : M \rightarrow \mathbb{R}^n$ we get $B_{ij} = \rho^{-1}(r_i - r)\delta_{ij}$. We define Laguerre shape operator

$$(6.18) \quad \mathbb{S} = \rho^{-1}(S^{-1} - r \text{id}) : TM \rightarrow TM.$$

Then \mathbb{S} is a self-adjoint operator with respect to the Laguerre metric $g = \rho^2 III$. It follows that for different principal radius r_i, r_j and r_k , the quotient $(r_i - r_j)/(r_i - r_k)$ is a Laguerre invariant.

In case $n = 3$, a complete Laguerre invariant system for surfaces in \mathbb{R}^3 is given by $\{g, \mathbb{B}, \mathbb{L}\}$.

§ 7. Laguerre minimal hypersurfaces in \mathbb{R}^n

In this section we calculate the first variation formula for Laguerre minimal hypersurfaces in \mathbb{R}^n .

Let $x_0 : M \rightarrow \mathbb{R}^n$ be a compact oriented hypersurface in \mathbb{R}^n with boundary ∂M . We assume that x_0 is umbilical free and its principal curvatures are non-zero. Let $x : M \times \mathbb{R} \rightarrow \mathbb{R}^n$ be a variation of x_0 , such that for each $t \in (-\varepsilon, \varepsilon)$ hypersurface $x_t = x(t, \cdot) : M \rightarrow \mathbb{R}^n$ is umbilical free and its principal curvatures are non-zero. Moreover, for any point $p \in \partial M$ we have

$$(7.1) \quad x_t(p) = x_0(p), \quad dx_t(p) = dx_0(p) : T_p M \rightarrow T_p M.$$

Then the Laguerre volume of x_t is given by

$$(7.2) \quad L(t) = L(x_t) = \int_M \frac{(\sum_i (r - r_i)^2)^{(n-1)/2}}{r_1 r_2 \cdots r_{n-1}} dM.$$

Our purpose is to calculate the derivative $L'(0)$.

Let $\{E_1, E_2, \dots, E_{n-1}\}$ be an orthonormal basis for the Laguerre metric g_t of x_t with dual basis $\{\omega_1, \omega_2, \dots, \omega_{n-1}\}$ for T^*M . We write the variation vector field of $x : M \times \mathbb{R} \rightarrow \mathbb{R}^n$ by

$$(7.3) \quad \frac{\partial x}{\partial t} = \sum_i u_i e_i(x) + \rho^{-1} f \xi,$$

where ξ is the unit normal of the hypersurface x_t and ρ the function defined by (6.1) for x_t . Then we have

$$(7.4) \quad \frac{\partial \xi}{\partial t} = \sum_i w_i E_i(x)$$

for some functions w_i . Since the second fundamental form is non-degenerate, we can write $E_i(x) = \sum_j A_{ij} E_j(\xi)$ with $\det(A_{ij}) \neq 0$. Let Y be the Laguerre position vector of x_t given by (6.1). Thus we have

$$(7.5) \quad (x \cdot E_i(x), -x \cdot E_i(x), E_i(x), 0) = 0 \mod\{Y, E_1(Y), \dots, E_{n-1}(Y)\}.$$

It follows from (7.3), (7.4) and (7.5) that

$$\frac{\partial Y}{\partial t} = \rho \left(\frac{\partial x}{\partial t} \cdot \xi, -\frac{\partial x}{\partial t} \cdot \xi, \mathbf{0}, 0 \right) = f \wp, \mod\{Y, E_1(Y), \dots, E_{n-1}(Y)\}.$$

Thus we can write

$$(7.6) \quad \frac{\partial Y}{\partial t} = \sigma Y + \sum_i v_i E_i(x) + f \wp$$

for some function σ and some tangent vector field $V = \sum_i v_i E_i$. We note that the function f is determined by the normal component of the variation vector field given by (7.3).

Let $\{Y, N, E_1(Y), \dots, E_{n-1}(Y), \eta, \wp\}$ be the Laguerre moving frame of x_t . Using the products of the frame we get

$$(7.7) \quad dY = \alpha Y + \sum_i \Omega_i E_i(Y) + \beta \wp;$$

$$(7.8) \quad dN = -\alpha N + \sum_i \Psi_i E_i(Y) + \gamma \wp;$$

$$(7.9) \quad dE_i(Y) = \Psi_i Y + \Omega_i N + \sum_j \Omega_{ij} E_j(Y) + \Phi_i \wp;$$

$$(7.10) \quad d\eta = -\gamma Y - \beta N + \sum_i \Phi_i E_i(Y),$$

where $\{\alpha, \beta, \Omega_i, \Omega_{ij}, \Psi_i, \Phi_i, \gamma\}$ are some 1-forms on $M \times \mathbb{R}$. From (6.6), (6.7), (6.8), (7.6) and the formula

$$d = \sum_i \omega_i E_i(x) + dt \frac{\partial}{\partial t} : C^\infty(M \times \mathbb{R}) \rightarrow \Lambda^1(M \times \mathbb{R})$$

we get

$$(7.11) \quad \alpha = \sigma dt; \quad \Omega_i = \omega_i + v_i dt, \quad \beta = f dt, \quad \Psi_i = \sum_j L_{ij} \omega_j + a_i dt;$$

$$(7.12) \quad \Phi_i = \sum_j B_{ij} \omega_j + b_i dt; \quad \Omega_{ij} = \omega_{ij} \omega_k + p_{ij} dt, \quad \gamma = \sum_i C_i \omega_i + c dt,$$

where a_i, b_i, c, p_{ij} are functions with $p_{ij} + p_{ji} = 0$ and ω_{ij} be the connection form of g_t .

Taking derivatives of (7.7), (7.8), (7.9) and (7.10) we get

$$(7.13) \quad d\beta - \sum_i \Omega_i \wedge \Phi_i - \alpha \wedge \beta = 0;$$

$$(7.14) \quad d\Omega_i - \sum_j \Omega_j \wedge \Omega_{ji} - \alpha \wedge \Omega_i = 0;$$

$$(7.15) \quad d\Phi_i - \sum_j \Omega_{ij} \wedge \Phi_j - \Psi_i \wedge \beta - \Omega_i \wedge \gamma = 0.$$

Since

$$d = d_M + dt \wedge \frac{\partial}{\partial t} : \Lambda^1(M \times \mathbb{R}) \rightarrow \Lambda^2(M \times \mathbb{R}),$$

where d_M is the differential operator on M , by comparing the coefficients of $\omega_i \wedge dt$ of (7.13) and (7.14) we get

$$(7.16) \quad b_i = E_i(f) + \sum_j B_{ij} v_j;$$

$$(7.17) \quad \frac{\partial \omega_i}{\partial t} = \sum_j (v_{i,j} + p_{ij} + \sigma \delta_{ij}) \omega_j;$$

where $\{v_{i,j}\}$ is the covariant derivative of $V = \sum_i v_i E_i$. By comparing the coefficients of $\omega_i \wedge dt$ of (7.15) and using (7.16), (7.17) we get

$$(7.18) \quad \frac{\partial B_{ij}}{\partial t} + \sigma B_{ij} = f_{i,j} + \sum_k v_k (B_{ki,j} + C_j \delta_{ik}) + \sum_k (p_{ik} B_{kj} - p_{jk} B_{ki}) - L_{ij} f - c \delta_{ij},$$

where $(f_{i,j})$ is the Hessian matrix of f . From (6.11), (6.13) and (6.14) we have

$$(7.19) \quad \sum_i B_{ii} = 0, \quad \sum_{ij} B_{ij}^2 = 1, \quad \sum_{ij} (B_{ki,j} + C_j \delta_{ik}) B_{ij} = 0.$$

By multiplying B_{ij} to (7.18), taking sum and using (7.19) we get

$$(7.20) \quad \sigma = \sum_{ij} (f_{i,j} B_{ij} - f L_{ij} B_{ij}).$$

Now we come to calculate $L'(0)$. Since the Laguerre volume can be written as

$$L(t) = \int_M \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{n-1},$$

we get from (7.17) that

$$L'(0) = \int_M (\operatorname{div} V + (n-1)\sigma) dM,$$

where $V = \sum_i v_i E_i$. Since on ∂M we have $v_i = 0$, $f = 0$ and $f_i = E_i(f) = 0$, it follows from (7.20) and Green-formula that

$$(7.21) \quad L'(0) = (n-1) \int_M \sum_{ij} (B_{ij,ij} - L_{ij} B_{ij}) f dM.$$

Thus we obtain

Theorem 7.1 *The first variation of a Laguerre volume for hypersurfaces in \mathbb{R}^n depends only on the normal component of the variation vector field. The Euler-Lagrange equation of the Laguerre functional is given by*

$$(7.22) \quad \sum_{ij} (B_{ij,ij} - L_{ij} B_{ij}) = 0.$$

Using (6.15) we can write the Euler-Lagrange equation by

$$(7.23) \quad \sum_i C_{i,i} - \frac{1}{n-2} \sum_{ij} L_{ij} B_{ij} = 0.$$

From (6.8), (6.7) and (6.15) we obtain

$$(7.24) \quad \Delta\eta = \sum_i (-C_{i,i} + \sum_j L_{ij} B_{ij}) Y + (n-3) \sum_i C_i E_i(Y) + \wp.$$

§ 8. Hypersurfaces in the Laguerre space forms

Using Laguerre embedding σ and τ in §4 we can regard a hypersurface $x : M \rightarrow \mathbb{R}_1^n$ or $x : M \rightarrow \mathbb{R}_0^n$ as a Laguerre hypersurface $(x', \xi') : M \rightarrow U\mathbb{R}^n$. In this section we study the relations between x and x' .

Let $x : M \rightarrow \mathbb{R}_1^n$ be a space-like oriented hypersurfaces in the Lorentzian space \mathbb{R}_1^n with the inner product \langle, \rangle given in (4.1). Let ξ be the normal of x with $\langle \xi, \xi \rangle = -1$. The shape operator $S : TM \rightarrow TM$ of x is defined by $d\xi = -dx \circ S$. Since S is self-adjoint on TM , all eigenvalues $\{k_i\}$ of S are real. We assume that $k_i \neq 0$. We define $r_i = 1/k_i$ the curvature radius of x and $r = (r_1 + r_2 + \cdots + r_{n-1})/(n-1)$ the mean curvature radius of x . Let e_i be the eigenvector of x with respect to the eigenvalue k_i . Then we have

$$(8.1) \quad e_i(x) = -r_i e_i(\xi).$$

In particular, we have $e_i(x_1) = -r_i e_i(\xi_1)$. We define $(x', \xi') = \sigma(x, \xi) : M \rightarrow U\mathbb{R}^n$, where $\sigma : U\mathbb{R}_1^n \rightarrow U\mathbb{R}^n$ is the Laguerre embedding given by (4.11). By a direct calculation we get from (4.11) and (8.1) that

$$(8.2) \quad e_i(x') = -(r_i \xi_1 + x_1) e_i(\xi').$$

It follows that e_i is the principal vector for the Laguerre hypersurface $f' = (x', \xi') : M \rightarrow U\mathbb{R}^n$ corresponding to the curvature radius

$$(8.3) \quad r'_i = r_i \xi_1 + x_1,$$

which implies the following relations between the mean curvature radius r' and r :

$$(8.4) \quad r' = r \xi_1 + x_1, \quad \rho'^2 = \sum_i (r'_i - r')^2 = \xi_1^2 \sum_i (r_i - r)^2 = \xi_1^2 \rho^2.$$

It is easy to verify from (4.11) that

$$(8.5) \quad Y' = \rho'(x' \cdot \xi', -x' \cdot \xi', \xi', 1) = \rho(\langle x, \xi \rangle, -\langle x, \xi \rangle, 1, \xi) = Y.$$

Thus the Laguerre metric is given by

$$(8.6) \quad g' = \rho'^2 III' = \langle dY', dY' \rangle = \langle dY, dY \rangle = \rho^2 III = g,$$

where III and III' are the third fundamental forms of x and x' , respectively. By (4.4) we know that the mean curvature radius sphere $H(x + r\xi, r)$ in $U\mathbb{R}_1^n$ corresponds to the vector $[\eta] \in \mathbb{Q}^{n+1}$, where

$$(8.8) \quad \eta = \left(\frac{1}{2}(1 + \langle x, x \rangle), \frac{1}{2}(1 - \langle x, x \rangle), 0, x\right) + r(\langle x, \xi \rangle, -\langle x, \xi \rangle, 1, \xi).$$

By a direct calculation we know that

$$(8.9) \quad \eta' = \left(\frac{1}{2}(1 + |x'|^2), \frac{1}{2}(1 - |x'|^2), x', 0\right) + r'(x' \cdot \xi', -x' \cdot \xi', \xi', 1) = \eta.$$

Thus the Laguerre embedding $\sigma : U\mathbb{R}_1^n \rightarrow U\mathbb{R}^n$ takes the mean curvature radius sphere $H(x + r\xi, r)$ in \mathbb{R}_1^n into the mean curvature radius sphere $S(x' + r'\xi', r')$ in \mathbb{R}^n .

Let $x : M \rightarrow \mathbb{R}_0^n$ be a space-like oriented hypersurfaces in the degenerate hyperplane \mathbb{R}_0^n of the Lorentzian space \mathbb{R}_1^{n+1} with the inner product \langle, \rangle given in (4.12). Let ξ be the unique vector satisfying

$$(8.10) \quad \langle \xi, dx \rangle = 0, \quad \langle \xi, \xi \rangle = 0, \quad \langle \xi, \nu \rangle = 1.$$

We define the shape operator $S : TM \rightarrow TM$ by $d\xi = -dx \circ S$. Since S is self-adjoint, all eigenvalues $\{k_i\}$ of S are real. We assume that $k_i \neq 0$. Then we define $r_i = 1/k_i$ the curvature radius of x and $r = (r_1 + r_2 + \cdots + r_{n-1})/(n-1)$ the mean curvature radius of x . Let $\tau : U\mathbb{R}_0^n \rightarrow U\mathbb{R}^n$ be the Laguerre embedding defined by (4.23) and $(x', \xi') = \tau \circ (x, \xi)$. By a similar way as we can show that

$$(8.11) \quad Y = \rho(\langle x, \xi \rangle, -\langle x, \xi \rangle, \xi) = \rho'(x' \cdot \xi', -x' \cdot \xi', \xi', 1) = Y',$$

where $\rho'^2 = \sum_i (r'_i - r')^2$ and $\rho^2 = \sum_i (r_i - r)^2$. Thus the Laguerre metric is given by

$$(8.12) \quad g = \rho^2 III = \rho'^2 III' = g',$$

where III and III' are the third fundamental forms of x and x' , respectively. Moreover, we have $\eta = \eta'$, where

$$(8.13) \quad \eta = \left(\frac{1}{2}(1 + \langle x, x \rangle), \frac{1}{2}(1 - \langle x, x \rangle), x\right) + r(\langle x, \xi \rangle, -\langle x, \xi \rangle, \xi);$$

$$(8.14) \quad \eta' = \left(\frac{1}{2}(1 + |x'|^2), \frac{1}{2}(1 - |x'|^2), x', 0\right) + r'(x' \cdot \xi', -x' \cdot \xi', \xi', 1).$$

It follows immediately from (6.1), (5.31), (8.5), (8.8), (8.11) and (8.13) that

Proposition 8.1 *Let x be a Laguerre hypersurface in $U\mathbb{R}^n$, $U\mathbb{R}_1^n$ or $U\mathbb{R}_0^n$. Let $\mathbf{c} \in \mathbb{R}_2^{n+3}$ be the time-like vector $\mathbf{c} = (0, 0, \mathbf{0}, -1)$, the space-like vector $\mathbf{c} = (0, 0, 1, \mathbf{0})$ or the light-like $\mathbf{c} = (0, 0, \nu)$, respectively. Let $\{r_i\}$ be the curvature radius of x . Let r the mean curvature radius of x and $\rho^2 = \sum_i (r_i - r)^2$. Then we have $\langle Y, \mathbf{c} \rangle = \rho$ and $\langle \eta, \mathbf{c} \rangle = r$.*

Theorem 8.1 *A surface in \mathbb{R}^3 , \mathbb{R}_1^3 or \mathbb{R}_0^3 (regarded as a Laguerre surface in \mathbb{R}^3) is Laguerre minimal if and only if $\Delta_{III} r = 0$.*

Proof. From (7.24) we know that for surface case we have

$$(8.15) \quad \Delta_{III} \eta = \rho^2 \sum_i (-C_{i,i} + \sum_j L_{ij} B_{ij}) Y + \rho^2 \wp.$$

Any surface in \mathbb{R}_1^3 and \mathbb{R}_0^3 can be regarded as surface in \mathbb{R}^3 with the same Y and η . Let $\mathbf{c} \in \mathbb{R}_2^{n+3}$ be the vector given in Proposition 8.1. Since $\langle \mathbf{c}, \wp \rangle = 0$, we get from (8.15) that

$$(8.16) \quad \Delta_{III} r = \rho^3 \sum_i (-C_{i,i} + \sum_j L_{ij} B_{ij}).$$

Thus we complete the proof of Theorem 8.1.

Remark. For Laguerre minimal surface in \mathbb{R}^3 the equation $\Delta_{III}r = 0$ is given in Blaschke's book [1].

Since $r = (k_1 + k_2)/k_1k_2$, we know that $r = 0$ if and only if x is a minimal surface in \mathbb{R}^3 , \mathbb{R}_1^3 or \mathbb{R}_0^3 . Thus a minimal surface in \mathbb{R}_1^3 or \mathbb{R}_0^3 induces a Laguerre minimal surface in $U\mathbb{R}^3$ (by using Laguerre embedding).

Theorem 8.2 *The only compact Laguerre minimal surface in \mathbb{R}^3 is the round sphere.*

Proof. Let $x : M \rightarrow \mathbb{R}^3$ be a compact Laguerre minimal surface. From (8.15) we $\Delta_{III}\eta = \rho^2\wp$, which holds also at umbilical points of x . Since $\eta = (\alpha, \beta, x + r\xi, r)$ for some functions α and β , we get

$$\Delta_{III}r = 0, \quad \Delta_{III}(x + r\xi) = 0.$$

Since M is compact, we know that both r and $x + r\xi = x_0$ are constant. Thus we have $|x - x_0|^2 = r^2$ and x is a round sphere in \mathbb{R}^3 . We complete the proof of Theorem 8.2.

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